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ON EXCITATION OF NORMAL AND ASSOCIATED WAVES IN AN INFINITE LAMINAR ELASTIC STRIP

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Forced harmonic vibrations in an infinite laminar elastic strip are considered. They are represented by the sum of normal and associated waves being propagated along the strip, i.e., along the layers. The properties of these waves, including the dispersion characteristics, are studied.

The expansion of forced vibrations of an elastic laminar strip in normal and associated waves travelling along the strip is performed here by the method of normal waves, (*) developed in [1– 3]. To apply it, the left side of the inhomogeneous boundary value problem for the amplitudes of the vibrations is represented as the sum of two differential operators, one depends only on the transverse coordinate of the strip y, and the other on the coordinate x along the strip. The "longitudinal" operator should be of first order. The initial boundary value problem for the displacement amplitudes is reduced to the above-mentioned canonic form by doubling the dimensionality of the vector-function on which the operators act. The desired representation is obtained by expanding the solution in a series of eigen- and associated functions of the transverse operator. Its eigenvalues are the wave numbers of the normal waves. The mentioned operator is nonself-adjoint, even in a lossless strip, hence, waves with complex wave numbers and waves associated to the normal waves enter into the solution in addition to the undamped normal waves.

Let us note that Lamb [4] obtain a solution for a homogeneous strip in 1904 in the form of the sum of waves travelling along the x-axis with constant phase velocities and invariant amplitude distribution modes of the displacements along the y-axis by the method of transforming the contour integral. It is shown in [2, 3] that these waves coincide with the normal waves. The associated waves were first introduced in [2] in the problem of electromagnetic wave propagation in a laminar medium.

The substitution method (see [5], say) in which the expression for the wave which retains the vibrations mode along the y-axis and the phase velocity along the x-axis, is substituted into the initial boundary value problem for the amplitudes of the elastic strip displacement, is also used extensively. It results in a spectral problem for a square beam [6 – 8], equivalent to the spectral problem for the transverse operator (see Sect. 3) and governing the wave numbers and shortened modes of the normal waves, but not the amplitudes:

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*) Krasnushkin, P. E., Method of normal waves in application to waveguides and their algebraic prototypes. Doct. Dissertation, Moscow State University, 1945.

1. Initial boundary value problem and its reduction to canonical form. Letting D denote partial derivatives with respect to the coordinates indicated in the subscript, we obtain the boundary value problem

$$
\begin{aligned}
&[E_0 D_{xx} + E_1 D_{yy} + E_2 D_{xy} + E_3 D_y + E_4 D_x + \rho\omega^2]\mathbf{u} = \mathbf{f} \\
&y = 0, \quad y = d, \quad [E_1 D_y + E D_x]\mathbf{u} = 0 \qquad (1.1) \\
&|x| \to \infty, \quad |\mathbf{u}(x, y)| \to 0 \\
&E_0 = \begin{Vmatrix} \nu_* & 0 \\ 0 & \mu_* \end{Vmatrix}, \quad E_1 = \begin{Vmatrix} \mu_* & 0 \\ 0 & \nu_* \end{Vmatrix}, \quad E_2 = \begin{Vmatrix} 0 & \nu_* - \mu_* \\ \nu_* - \mu_* & 0 \end{Vmatrix}, \\
&E = \begin{Vmatrix} 0 & \mu_* \\ \lambda_* & 0 \end{Vmatrix} \\
&E_3 = D_y E_1, \quad E_4 = D_y E \\
&\nu_* = \lambda_* + 2\mu_*, \quad \lambda_* = \lambda + i\omega(\eta - 2\zeta / 3), \; \mu_* = \mu + i\omega\zeta
\end{aligned}
$$

for the vector-function $\mathbf{u} = \mathrm{col}\,(u_x, u_y)$, where u_x and u_y are the displacement amplitudes of the vibrations in the strip $(-\infty < x < \infty, \; 0 \leqslant y \leqslant d)$ caused by the force $\mathbf{f}(x, y) \exp i\omega t$. Here, the Lamé coefficients λ, μ and the strip density ρ are continuous functions of y, while η and ζ are viscosity coefficients which vanish when the dissipation parameter $\varepsilon \to 0$. The boundary conditions for $y = 0$ and $y = d$ in (1.1) refer to a free strip. For strips clamped at the edges, they are replaced by the conditions $\mathbf{u} = 0$. The function $\mathbf{f}(x, y) = \mathrm{col}\,(f_x, f_y)$ differs from zero only in the interval (x_1, x_2). For $\varepsilon \neq 0$ the boundary value problem (1.1) has a unique solution. The solution for a lossless strip ($\eta = \zeta = 0$) is obtained therefrom for $\varepsilon \to 0$.

To reduce the boundary value problem (1.1) to canonical form, we introduce the vector function $\mathbf{v} = \mathrm{col}\,(v_x, v_y)$ by using the relationship (1.2)

$$
\mathbf{v} = [i\alpha D_x + \beta D_y]\mathbf{u}; \quad \alpha = \|\alpha_{mn}\|, \quad \beta = \|\beta_{mn}\|; \quad m, \; n = 1, 2 \qquad (1.2)
$$

Here α_{mn}, β_{mn} are arbitrary complex numbers subject to the condition $\det \alpha \neq 0$. From (1.1) we obtain the boundary value problem for the four-component vector function $\mathbf{w} = \mathrm{col}\,(u_x, u_y, v_x, v_y)$ in the canonical form used in [1 – 3]

$$
\begin{aligned}
&\mathbf{L}_x \mathbf{w} + \mathbf{L}_y \mathbf{w} = \mathbf{F}(x, y) = \mathrm{col}\,(\mathbf{0}, \mathbf{f}^0) \qquad (1.3) \\
&\mathbf{0} = \mathrm{col}\,(0, 0), \quad \mathbf{f}^0 = \mathrm{col}\,(f_x^0, f_y^0) = -\alpha E_0 \,\mathrm{col}\,(f_x, f_y)
\end{aligned}
$$

Here $\mathbf{L}_x$ is a first order differential operator generated by the differential expression $l_x = i\mathbf{E}\partial / \partial x$ ($\mathbf{E}$ is the unit matrix of dimension 4×4) and the boundary conditions $|\mathbf{w}| \to 0$ as $|x| \to \infty$, $\mathbf{L}_y$ is the "transverse" differential operator generated by the block matrix of the differential expressions

$$
\begin{aligned}
&l_y = \|l_{ij}\|; \; i, j = 1, 2; \; l_{11} = BD_y, \; l_{12} = -\alpha^{-1}, \; B = \alpha^{-1}\beta \qquad (1.4) \\
&l_{21} = \alpha(B^2 - iE_2'B - E_1')D_{yy} - \alpha(E_3' + iE_4'B)D_y - \\
&\quad \rho\omega^2\alpha E_0^{-1} \\
&l_{22} = (i\alpha E_2' - \beta)\alpha^{-1}D_y + i\alpha E_4'\alpha^{-1}; \quad E_k' = E_0^{-1}E_k, \\
&k = 1, 2, 3, 4
\end{aligned}
$$

and the boundary conditions

$$y = 0,\ y = d,\quad [E_1 + iEB]D_y \text{ and } - iE\alpha^{-1}\mathrm{v} = 0 \qquad (1.5)$$

The operator L_y in whose eigen-elements the solution of the problem (1. 3) is expanded, is obtained from the operator $\mathbf{L}_y$ by replacing $\partial / \partial y$ by d / dy. The operator L_y acts on the vector function $\mathbf{W}(y) = \mathrm{col}\,(U_x, U_y, V_x, V_y) = \mathbf{w}$, where x is considered a fixed parameter. Here $\mathbf{W}(y)$ is an element of the functional vector space with the scalar product $(\mathbf{W}, \mathbf{W}')$ from $\mathbf{L}_2$.

The spectral properties of L_y are independent of the selection of α and β (see Sect. 3 below). In order to give physical meaning to the scalar product, we select the specific operator L_y^1, $\alpha_{11} = \lambda_* + 2\mu_*$, $\alpha_{22} = -\mu_*$, $\beta_{12} = i\lambda_*$, $\beta_{21} = -i\mu_*$ as the operator L_y (the remaining α_{mn} and β_{mn} equal zero). Then the vector-function $\mathbf{W}(x, y) = \mathrm{col}\,(u_x, u_y, i\sigma_{xx}, -i\sigma_{xy})$, where σ_{xx}, σ_{xy} are components of the stress tensor σ. The flux of vibrational power P through a given section, averaged with respect to time and the section $x = \mathrm{const}$, is expressed in terms of the scalar product

$$P = (\mathbf{J}w, w) = \int_0^d [(u_x\sigma_{xx}^* - u_y\sigma_{xy}^*) + (u_x{}^*\sigma_{xx} - u_y{}^*\sigma_{xy})]\,dy, \qquad (1.6)$$

$$J = i\begin{pmatrix} 0 & -\mathbf{E} \\ \mathbf{E}, & 0 \end{pmatrix}$$

Here $\mathbf{E}$ is a unit matrix of dimension 2×2. J is the J-operator, and the asterisk denotes the complex conjugate values.

Differentiating (1. 6) and taking (1. 3) into account, we obtain the relation

$$D_yP = i\,(JL_y{}^1\mathbf{w}, \mathbf{w}) - i\,(\mathbf{w}, JL_y{}^1\mathbf{w}) \qquad (1.7)$$

in the intervals of the x-axis where there are no forces.

Because of the boundary condition for $|x| \to \infty$ in the case $\varepsilon \neq 0$ $D_xP < 0$ for $x > x_2$ and $D_xP > 0$ for $x < x_1$. For $\varepsilon = 0$, it follows from the energy conservation law that $D_xP = 0$ and $(JL_y{}^1\mathbf{w}, \mathbf{w}) = (\mathbf{w}, JL_y{}^1\mathbf{w})$, i.e., the operator $L_y{}^1$ is J- self-adjoint.

2. Expansion of the solution of the boundary value problem in normal and associated waves travelling along a strip. It is shown below (see Sect. 3) that the spectrum of the operator $L_y{}^1$ is discrete with a single condensation point at infinity. Moreover, the operator $L_y{}^1$ is Tamarkin regular [9] (*). Hence, by assuming the function $\mathbf{F}$ to be sufficiently smooth in y, we represent the solution (1. 3) in the form

$$\mathbf{w}(x, y) = \sum_j \sum_{k=1}^{\tau_j} \Psi_j{}^k(x)\,\mathbf{W}_j{}^k(y) \qquad (2.1)$$

*) A. G. Kostiuchenko turned the author's attention to this fact.

Here W_j^k $(k = 1, 2, \ldots, \tau_j)$ are the eigen $(k = 1)$ and associated $(k > 1)$ functions of the operator L_y^1 determined from the equation

$$(L_y^1 - \gamma_j E) W_j^k + W_j^{k-1} = 0, \quad W_j^\circ = 0 \qquad (2.2)$$

Here γ_j are eigenvalues of the operator L_y^1. The functions W_j^k are subject to the biorthogonality conditions

$$(W_j^k, G_{j'}^{k'}) = 0, \ j \neq j' \quad \text{or} \quad k \neq k'; \ (W_j^k, G_j^k) = N_j^k \qquad (2.3)$$

where G_j^k are the eigen and associated functions of the operator conjugate to L_y^1. The functions G_j^k are determined from a chain of equations analogous to (2.2) [3]. Because of the J - self-adjointness of L_y^1, the scalar products in (2.3) are expressed in terms of an integral of the power flux P (1.6). This does not hold for arbitrary α and β. Thus, for instance, in the case of [10] which follows from (1.2) for $\alpha_{11} = \beta_{12} = \lambda_* + 2\mu_*$, $\alpha_{22} = \beta_{21} = -\mu_*$ (the remaining α_{mn} and β_{mn} are zero), the so-called weighted orthogonality occurs (the integrals are supplemented by sums from the boundary conditions).

Substituting (2.1) into (1.3) and taking account of (2.3), we obtain a chain of first order equations for $\Psi_j^k (k = 1, 2, \ldots, \tau_j)$

$$i d\Psi_j^k / dx + \gamma_j \Psi_j^k + \Psi_j^{k+1} = (F, G_j^k) / N_j^k, \quad \Psi_j^{\tau_j+1} \equiv 0, \qquad (2.4)$$
$$|x| \to \infty, \quad |\Psi_j^k| \to 0$$

We seek the solution (2.4) for a concentrated force $F(x, y)\delta(x - x')$ by using a Green's function. Consequently, the expansion (2.1) becomes

$$w(x, x'; y) = \sum_{j\pm} \sum_{l=1}^{\tau_{j\pm}} F_{j\pm}^l (x') Q_{j\pm}^l (x, x'; \gamma_{j\pm} \qquad (2.5)$$

$$Q_{j\pm}^l (x, x'; \gamma_{j\pm}) = \exp[i\gamma_{j\pm}(x - x')] \Big[W_{j\pm}^l + i(x - x') W_{j\pm}^{l-1} + \ldots + \frac{i^{l-1}(x - x')^{l-1}}{(l-1)!} W_{j\pm}^1 \Big]$$

Here $F_j^l(x')$ are the right sides of (2.4), the γ_j in the upper and lower half-planes γ are marked by the plus and minus signs (for $\varepsilon \neq 0$ there are no real γ_j) ; $w(x, x'; y)$ equals the right side of (2.5) with the plus sign for $x > x'$ and the minus sign for $x < x'$.

By definition, the term of the sum (2.5) of number j $(l = 1)$ is called the mode of the normal wave of number j. A normal wave corresponds to each simple eigenvalue γ_j, which differs from zero on the right $(x > x')$ for γ_{j+} and left $(x < x')$ for γ_{j-} of the section $x = x'$. The four-component vector function $W_j^1(y)$ is called the mode of the normal wave of number j. It follows from (1.2) that

$$W_j^1(y) = \begin{Vmatrix} E \\ Z_j \end{Vmatrix} U_j^1(y); \quad Z_j = i\gamma_j \begin{Vmatrix} \lambda_* + 2\mu_* & 0 \\ 0 & -\mu_* \end{Vmatrix} + i \begin{Vmatrix} 0 & \lambda_* \\ -\mu_* & 0 \end{Vmatrix} D_y \qquad (2.6)$$

Here $\mathbf{Z}_j$ is the wave resistance operator of the normal wave of number j which sets up a relationship between the stress-tensor and displacement components in the vector-function of the normal wave mode. The normal wave can be reproduced in the whole range (x', ∞) or $(-\infty, x')$ in a certain section of the strip $x' =$ const according to the given mode of $\mathbf{W}_j^1$, which permits considering the normal wave as an evolutionary process being developed along the x axis.

According to [2,3], the terms of the sum (2.5) with numbers $j, l\,(l > 1)$ are called associated waves. Characteristic for them is the growth of the amplitude according to a power law with the increase in $|x - x'|$, which is suppressed by exponential damping always available for $\varepsilon \neq 0$, that assures compliance with the condition (1.1) as $|x| \to \infty$.

Integrating (2.5), we obtain the solution of the problem (1.3) for an arbitrary external force $\mathbf{F}\,(x, y)$ in the form

$$\mathbf{w}(x, y) = \sum_{j+} \sum_{l=1}^{\tau_{j+}} \int_{-\infty}^{x} \mathbf{F}_{j+}^l(x')\, \mathbf{Q}_{j+}^l\, dx' + \sum_{j-} \sum_{l=1}^{\tau_{j-}} \int_{x}^{\infty} \mathbf{F}_{j-}^l(x')\, \mathbf{Q}_{j-}^l\, dx' \qquad (2.7)$$

3. Fundamental properties of normal waves. Eliminating $\mathbf{V}_j^1$ from (2.2) for $k = 1$, we obtain the spectral problem for a quadratic bundle in the parameter γ by taking (1.4) and (1.5) into account

$$[-\gamma^2 E_0 + i\gamma\,(E_2 D_y + E_4) + (E_1 D_{yy} + E_3 + \rho\omega^2)]\mathbf{U}^1 = 0 \qquad (3.1)$$
$$y = 0,\ y = d, \quad [E_1 D_y + i\gamma E]\mathbf{U}^1 = 0$$

The problem (3.1) determines the wave numbers γ_j and the shortened forms $\mathbf{U}_j^1$ of the normal waves. By eliminating $\mathbf{V}_j^k$ from the remaining equations in (2.2), we analogously obtain a chain of quadratic bundles to determine $\mathbf{U}_j^k$. Since α and β do not enter into the quadratic bundle equations, the following property then holds for the class of representations of the operator $L_y\,[\alpha, \beta]$.

1°. The wave numbers γ_j *of the normal waves are independent of the selection of* α *and* β *in the operator* L_y, and their modes differ only by the components

$$\mathbf{V}_j^1\,(y) = (-\gamma_j\alpha + \beta D_y)\mathbf{U}_j^1\,(y)$$

The quadratic bundles in [6— 8] are obtained from the initial equations by substituting $\mathbf{u} = \mathbf{U} \exp i\gamma x$, and the operator $L^0\,[\alpha = 1, \beta = 0]$ equivalent to the bundle is constructed for their investigation by doubling the dimensionality of $\mathbf{U}$ (but not of $\mathbf{u}$). Hence, by virtue of property 1°, the results in [8] about the vibrations of an elastic cylinder can be used. We extract the following property from among them.

2°. The spectrum of the wave numbers γ_j for a free and a clamped strip of variable density $\rho\,(y)$ is discrete for $\varepsilon = 0$ with a single condensation point at infinity. The number of real γ_j is finite and equal to $N\,(\varepsilon)$. The system of modes $\{\mathbf{W}_j^k\}$ is complete in $\mathbf{H}_2^1 \oplus \mathbf{H}_2^1$, and the system of shortened modes $\{\mathbf{U}_j^k\}$ is doubly complete in $\mathbf{H}_2^1$, where $\mathbf{H}_2^1$ is the Sobolev space.

The property 3° follows from the form of L_y^0 obtained from (1.4).

3°. For arbitrary ε the normal waves of the strip form pairs with the wave numbers γ_j and $-\gamma_j$.

4°. For $\varepsilon = 0$ the complex wave numbers form quartets, symmetric with respect to the real and imaginary axes of the γ plane. The modes $\mathbf{W}_{j\pm}$ of these normal waves for the operator L_y have the form $J\mathbf{G}_{j\mp}$.

Because of the J - self-adjointness of the operator $L_y{}^1$ for $\varepsilon = 0$, its spectrum forms a complex conjugate pair γ_j and $\gamma_j{}^*$, which taking the property 3° into account proves the property 4°.

5°. For $\varepsilon = 0$ the normal waves with imaginary and complex wave numbers do not carry wave energy along the strip.

There follows from property 4°: $P_{j+} = (J\mathbf{W}_{j+}, \mathbf{W}_{j+}) = (J\mathbf{W}_{j+}, J\mathbf{G}_{j-}) = 0$, which indeed proves property 5°.

Let us note that for waves with complex wave numbers there are energy fluxes different from zero in opposite directions in individual parts of the section $x = \text{const}$, which compensate each other on the average in the section.

From the conservation of energy law applied to a segment $(x, x + dx)$ of the strip, it follows that for small ε for normal waves with imaginary and complex wave numbers $P_j > 0$ for γ_{j+} and $P_j < 0$ for γ_{j-}. For normal waves with real γ_j, $\mathbf{W}_j{}^1 = J\mathbf{G}_j{}^1$ and $P_j = (J\mathbf{W}_j{}^1, \mathbf{W}_j{}^1) \neq 0$ for $\varepsilon = 0$. It follows from (1.7) that $P_j = P_0 \exp[-2\mathrm{Im}\gamma_j x]$. Hence, for small ε, by virtue of the energy conservation law for the above-mentioned segment, the normal waves with $P_j > 0$ will have the wave numbers γ_{j+} and γ_{j-} of normal waves with $P_j < 0$.

Below, we shall designate normal waves with the wave numbers γ_{j+} $(P_j > 0)$ as plus waves and normal waves with the wave numbers γ_{j-} $(P_j < 0)$ as minus waves. We shall also distinguish normal waves according to the dispersion sign, i.e., according to the group velocity direction $v_g = d\omega/d\gamma$. Since v_g equals the energy flux velocity, then the following property holds for normal waves with real γ_j.

6°. For $x > x'$ $(x < x')$ normal waves with positive dispersion have a phase being propagated in the positive (negative) direction of the x-axis i.e., along the flux P, and waves with negative dispersion have a phase being propagated from ∞ $(-\infty)$, i.e., opposite to the flux P.

The expression $\mathbf{u} = \mathbf{U}(y) \exp i\gamma x$ is substituted in the initial equations (1.1) in a number of papers studying the properties of waves being propagated along a strip without distortion of the mode, and this results in the problem (3.1) to determine γ_j and $\mathbf{U}_j(y)$. Since the problem (3.1) is equivalent to the spectral problem (2.2) for the operator L_y for $k = 1$, then all waves obtained by the substitution method, particularly Lamb waves as well as Rayleigh waves and other surface waves, are normal waves of the discrete spectrum of the operator L_y. Restoration of the complete modes $\mathbf{W}_j{}^1$ of these waves by means of the shortened modes $\mathbf{U}_j{}^1$ obtained is performed by means of (2.6) and contains arbitrariness in the selection of α and β. Waves in anisotropic strips are studied by the substitution method in [11].

4. Dispersion dependences for normal waves. Let ω, d denote the coefficients of the parametrized functions $\rho(y)$, $\lambda(y)$ and $\mu(y)$ in terms of $p_1, p_2, \ldots, p_i, \ldots, p_R$ or, briefly, $\mathbf{p} = \{p_i\}$. Let us consider them complex quantities, and $\mathbf{p}$ a point of parametric space $\mathbf{C}^R$ of the complex variables of dimensionality R. The dependences $\gamma_i(\mathbf{p})$ obtained from (2.2) for $k = 1$, where $L_y = L_y(\mathbf{p})$ are called dispersion relations when the point $\mathbf{p} \in \mathbf{C}^R$ performs a certain path Λ in $\mathbf{C}^R$. Exactly as in [12, 13], we consider them branches of the analytic function R of the complex variables which describe wave conversion in the neighborhoods of branch points which are points of multiplicity of γ_j.

Introducing the normalized basis $\{\mathbf{W}_r^0(y)\}$, $(\mathbf{W}_r^0, \mathbf{W}_s^0) = \delta_{rs}$ (δ_{rs} is the Kronecker delta) and expanding the function $\mathbf{w}$ therein, we go from (1.3) to a system of ordinary differential equations (C_r are coefficients of the expansion)

$$[i\mathbf{E}d/dx + \mathbf{A}]\,C_r = F_r, \quad \mathbf{A} = \| a_{rs} \|, \quad a_{rs} = (L_y\mathbf{W}_r^0, \mathbf{W}_s^0) \qquad (4.1)$$

which describes the set of interacting partial systems [14] obtained from (4.1) for $a_{rs} = 0$, $r \neq s$ and characterized by partial waves with the wave numbers a_{rr}. In the set of such systems normal waves, whose wave numbers are eigenvalues of the matrix $\mathbf{A}$, are possible for $a_{rs} \neq 0$. These waves are representations of the normal waves considered above in the basis $\{\mathbf{W}_r^0\}$. In conformity with (4.1), the coefficients a_{rs} depend on $\{p_i\}$ and we introduce the parameters $\{a_i\}$, formed by using algebraic operations on elements of the matrix $\mathbf{A}$ and which are in one-to-one correspondence with the parameters $\{p_i\}$ of the operator L_r, to study the dispersion dependences of normal waves in the representation of the basis $\{\mathbf{W}_r^0\}$. To study the neighborhood of the double multiplicity γ_1, γ_2 it is sufficient to consider the parametric space $\mathbf{C}^2(p_1, p_2)$. Let us replace its space $\mathbf{C}^2(a_1, a_2)$ by selecting $\mathbf{W}_r^0$, $r = 1, 2$, so that for all $(p_1, p_2) \in \mathbf{C}^2$ the influence of other partial systems on the system $r = 1, 2$ would be negligible in the neighborhood of double multiplicity points γ_1, γ_2, i.e., the coefficients of wave affinity $K_{rs} = \sqrt{a_{rs}a_{sr}}/(a_{rs} - a_{sr}) \ll 1$ for $r = 1, 2$ and any $r \neq s$. Let us put $a_1 = a_{11} - a_{22} = C\Delta\omega$, where $\Delta\omega = \omega - \omega_0$, $a_2 = Cl$, where $l = \sqrt{a_{12}, a_{21}}$ and C is a constant. Let us consider $p_1 = \omega$ and p_2 selected so that a one-to-one dependence exists between (p_1, p_2) and (a_1, a_2). In this case

$$\gamma_{1,2} \sim A + B\Delta\omega \pm C\sqrt{\Delta\omega^2 + l^2} \qquad (4.2)$$

and the modes of the normal waves in the basis $\mathbf{W}_r^0$, $r = 1, 2$ have the form

$$\mathbf{W}_{1,2}^1 \sim d_{1,2}^1\mathbf{W}_1^0 + d_{1,2}^2\mathbf{W}_2^0, \quad \varkappa_{1,2} \sim [\Delta\omega \mp \sqrt{\Delta\omega^2 + l^2}]\,/\,l \qquad (4.3)$$

Here $\varkappa = d^1 / d^2$ is the coefficent of the wave mode distribution [14] and $p_0 = p$ $(\Delta\omega = 0,\ l = 0) \in \mathbf{C}^2$, where $\gamma_1 = \gamma_2 = \gamma_0$, is a point of diagonalized multiplicity. Points of Jordan multiplicity $\tau_j = 2$ lie on the lines $\Delta\omega = \pm il$, which intersect at the point $p_0 \in \mathbf{C}^2$.

Let us construct a single-valued analytic function from (4.2). To do this [12, 13], we introduce two specimens of the space $\mathbf{C}^2$, $\mathbf{C}_1^2$ and $\mathbf{C}_2^2$, and we connect them by means of a hypersurface of a slit S_c passing through the lines $\Delta\omega_{1,2} = \pm il$. Conversion of normal waves into each other occurs on paths Λ intersecting S_c.

For cases when ω is real and the normal waves are subjected to the properties 3° and 4°, we consider a family of paths of two kinds: 1) l is imaginary and the frequencies $\omega_{1,2} = \omega_0 \pm |l|$ at which Jordan multiplicities occur, lie on the real ω axis; 2) l is real and the frequencies $\omega_{1,2} = \omega_0 \pm i|l|$ are complex-conjugate.

For imaginary l (conversion of the first kind), the paths Λ intersect both Jordan points. Taking account of property 4°, we obtain from (4.2) in the neighborhood of the first Jordan point $\gamma_{I}(\omega_1)$ $(\omega_1 = \omega_0 - |l|)$

$$\gamma_{1,2} \approx A + B(\omega - \omega_0) \pm iC\sqrt{2|l|(\omega - \omega_1)} \tag{4.4}$$

Here A, B and C are real, $|B| > |C|$. For $\omega < \omega_1$ the wave numbers γ_1 and γ_2 are real (undamped normal waves). Upon passing through $\omega = \omega_1$, the undamped normal waves are converted into waves with complex γ_1 and γ_2 (complex normal waves). The case $\omega = \omega_1$ is considered below in Sect. 5. For $\omega = \omega_2$ the reverse conversion of complex into undamped waves occurs $(\gamma_1 = \gamma_2 = \gamma_{II})$.

The frequencies ω_1 and ω_2 separate the pass and forbidden bands of normal waves with numbers $j = 1, 2$. For $|l| \to 0$ the forbidden band is narrowed, transforming into a point. At the multiplicity points γ_I and γ_{II} the condition $v_g = d\omega / d\gamma = 0$ is satisfied. The wave numbers of normal waves with different dispersion laws merge in them.

The dispersion dependences (4.4) were obtained by a number of authors for Lamb waves in homogeneous strips and plates (see [15], for instance) and for flexural waves in thin strips in [16].

Such conversions are encountered most often for $\gamma_I = 0$ or $\gamma_{II} = 0$, when the undamped waves are transformed into damped waves with imaginary wave numbers γ_1 and γ_2. In this case $A = B = \omega_0 = 0$ and $\omega_1 = \omega_2 = l$. Examples of such conversions can be found in Fig. 17 in [15]. Rayleigh already knew of them in acoustics and electrodynamics, when he studied waveguide effects in hollow tubes.

For real l let Λ intersect S_c between lines on which points of Jordan multiplicity lie (see Fig. 3 in [13]). In this case, A, B and C are real, B and C have the same sign, and $B > C$. Therefore, $d\omega / d\gamma$ does not equal zero in the neighborhood of a double multiplicity and the normal waves are undamped with dispersions of one sign. As the frequency ω approaches ω_0, the normal waves with γ_1 and γ_2 lose localization in the partial systems according to (4.3), and upon going through $\omega = \omega_0$ exchange modes. This effect, called a conversion of the second kind in [12], was first described in [14], and studied in [17] in the example of two waveguides coupled by a longitudinal slot. In homogeneous strips (plates) such conversions should be observed in the neighborhoods of the intersection of the partial system dispersion curves (see Fig. 17 in [15], say, where the conversion of waves of transverse and longitudinal polarization occurs at the intersection of the curves mentioned). Another example of a conversion of the second kind is considered in [18], where the role of the partial systems is performed by the Rayleigh wave and one of the Lamb waves localized in the unique waveguide originating because of wave refraction in a laminar inhomogeneous medium.

The global patterns of the dispersion curves (see Figs. 17 and 18 in [15], for instance) is quite complex although it consists locally of just the two conversions described above (with the exception of the point $\gamma = 0$). Let us note that the global classification of normal waves by means of the continuity of the curves $\gamma_j(\omega)$ is impossible, as has been shown in [13].

5. General pattern of forced vibrations of an elastic strip. According to (2.7), the field of forced vibrations depends on both the strip parameters, including the frequency ω, and the mode of the external force $\mathbf{F}(x, y)$. Normal waves with imaginary and complex wave numbers are localized near the source of the external force and do not carry wave energy away from it if ε is sufficiently small. They produce a wattless load at the source. Associated waves with non-real wave numbers for $\varepsilon = 0$ possess the same property. Let us note that associated waves with real γ_j do not occur. In order to show this, we turn to the case studied in Sect. 4. For instance, we consider the approach of ω to the critical frequency ω_1 when the origination of the associated wave should be expected since the wave numbers γ_1 and γ_2 are close. To do this, normal waves with the mentioned wave numbers should be excited on one side of the section $x = x'$. However, one of these is a plus-wave, and the other is a minus-wave. Hence, they are excited on different sides of the section $x = x'$, i.e., interference of the space beat type, which is necessary for the origination of an associated wave is excluded. But by virtue of property 3°, a point $-\gamma_I$ exists in addition to the point γ_I and for $\omega = \omega_1$ the interference between waves with the wave numbers γ_{1+} and γ_{2+} results in the standing wave

$$\text{const}\cos(\gamma_I x)\exp(i\omega_1 t) \tag{5.1}$$

If $\gamma_I = 0$, then the period of the standing wave becomes infinite, and a field homogeneous in x holds, which decreases exponentially for $\varepsilon \neq 0$ as $|x - x'|$ grows.

Furthermore, according to (2.7) and property 2°, the field of forced vibrations consists of a finite number of undamped normal waves. For $x > x_2$ it has the form

$$\mathbf{w}(x, y) \simeq \sum_{j+}^{N} C_{j+} \left\| \begin{matrix} \mathbf{E} \\ \mathbf{Z}_{j+} \end{matrix} \right\| \mathbf{U}_j^1(y)\exp\gamma_{j+}x \tag{5.2}$$

$$C_{j+} = \int_{x_1}^{x_2} F_{j+}^1(x')\exp[-i\gamma_{j+}x']\,dx'$$

For $x < x_1$ the plus sign in the subscripts should be replaced by a minus. There are no associated waves in (5.2). However, terms similar to (5.1) occur as approaches the critical values in (5.2). Each passage of ω through the critical value is accompanied by the occurrence or disappearance of one of the undamped normal waves, which results in a discontinuity in the derivative $\partial\mathbf{w}/\partial\omega$ in the dependence of the field $\mathbf{w}$ on the frequency ω.

The wave numbers of a pair of waves in (5.2) come together as the frequency ω passes through the domain of a conversion of the second kind. If this pair is dominant in the far field, then because of the space beats a periodic change in the polarizations will be observed with progress along the x-axis, as for instance, in the cases of [15] considered above, or alternate incidence of the Rayleigh and Lamp waves as, for instance, in the case of [18].

If f F exp ipx in the interval (x_1, x_2), then

$$\mathbf{w}(x, y) \approx \sum_{j}^{N} \frac{F_j^1}{p-\gamma_j} \left\| \begin{matrix} \mathbf{E} \\ \mathbf{Z}_j \end{matrix} \right\| \mathbf{U}_j^1 \exp[\gamma_j x - (p-\gamma_j)(x_2 - x_1)] \tag{5.3}$$

The summation in (5.3) is over $j+$ for $x > x_2$ and over $j-$ for $x < x_1$. For p close to $\operatorname{Re}\gamma_r$ and small $\operatorname{Im}\gamma_r$ (r is the number of one of the normal waves in the sum (5.3), the wave of number r will be dominant in the far field for $x > x_2$ for positive, and for $x < x_1$ for negative dispersion. This occurs because of the wave resonance between the external force wave $\mathbf{f} = \mathbf{F} \exp ipx$, and the normal wave of number r [19] which causes a linear growth of the normal wave amplitude in the interval (x_1, x_2). This phenomenon is used to excite individual types of normal waves [20].

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